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Stability of thermodynamic and dynamical order in a system of globally coupled rotors

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Abstract
A system of globally coupled rotors is studied in a unified framework of microcanonical and canonical ensembles. We consider the Fokker–Planck equation governing the time evolution of the system, and examine various stationary and non-stationary solutions. The canonical distribution, describing equilibrium, provides a stationary solution also in the microcanonical ensemble, which leads to order in a system with ferromagnetic coupling at low temperatures. On the other hand, the microcanonical ensemble admits additional stationary and non-stationary solutions; the latter allows dynamical order, characterized by multiple degrees of clustering, for both ferromagnetic and antiferromagnetic interactions. We present a detailed stability analysis of these solutions: in a ferromagnetic system, the canonical distribution is observed stable down to a certain temperature, which tends to get lower as the number of Fourier components of the perturbed distribution is increased in the analysis. The non-stationary solution remains neutrally stable below the critical temperature, indicating inequivalence between the two ensembles. For antiferromagnetic systems, all solutions are found to be neutrally stable at all temperatures, suggesting that dynamical ordering is relatively easy to observe at low temperatures compared with ferromagnetic systems.

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1. Introduction

The system of sinusoidally coupled oscillators serves as a prototype model describing various oscillatory phenomena in nature. When the coupling is short-range, i.e., between nearest neighbours, the oscillator system describes an array of Josephson junctions, which has been a subject of extensive studies [1]. On the other hand, there are also many systems with long-range couplings in physics and biology. Physiological rhythmic processes may be examples of the latter, which may be modelled as a system of coupled oscillators with the range of coupling being varied, where phase synchronization of the system is an important issue to be understood [2]. Physical examples are diverse, ranging from self-gravitating and plasma systems, where the long-range nature of the gravitational or Coulombic interaction gives rise to difficulty in understanding the systems. A system of globally coupled rotors has thus been proposed and studied to simulate those systems [3]. Here the interaction range is infinite, with the strength scaled with the system size, making the system of the mean-field character amenable to analytical treatment. In spite of the mean-field nature, however, the system has turned out to exhibit rich features in dynamical and statistical properties.

In the canonical ensemble one can find an analytic solution and the system with the ferromagnetic interaction undergoes an equilibrium phase transition at a finite critical temperature, whereas there is no phase transition for the antiferromagnetic interaction. On the other hand, direct simulations in the microcanonical ensemble reveal some interesting features with remarkable differences with the nature of the interaction. Specifically, for the ferromagnetic interaction, the system displays extremely slow relaxation towards the thermodynamic equilibrium. This slow relaxation, dubbed quasi-stationarity, does not coincide with predictions in the canonical ensemble, and thus the suggestion has been made that there may exist inequivalence between canonical and microcanonical ensembles. Such quasi-stationarity is observed to survive well below the equilibrium critical temperature and hence has attracted much attention [3–11], together with some controversy [7]. In the regime showing quasi-stationarity it has also been reported that the system exhibits ageing effects and glassy behaviour [7, 8]. For the antiferromagnetic interaction the system exhibits a different type of coherent motion at low temperatures, again only in the microcanonical ensemble [12]: the rotors move in two groups, called the bi-cluster, for a long time, which is explained in terms of the statistical equilibrium of the effective Hamiltonian obtained after averaging out fast variables.

In a recent work we have employed a novel approach that treats the system in a unified framework of microcanonical and canonical ensembles [13]. Starting from the set of Langevin equations describing dissipative dynamics of a system (canonical ensemble) and the corresponding Fokker–Planck equation (FPE), we have pointed out that the nondissipative Hamiltonian dynamics (microcanonical ensemble) may be described as a limiting case of the vanishing damping coefficient. Thereupon we have been able to find a class of solutions for the incoherent phase depending on the ensemble, some of which are neutrally stable even below the equilibrium critical temperature. This neutral stability has then been suggested to be a plausible physical explanation as to the origin of the quasi-stationarity observed in numerical experiments. In this paper we further extend the stability analysis of the previous work to the ferromagnetic coherent phase (with thermodynamic order) and to the system with the antiferromagnetic interaction. For the latter, we attempt to provide an alternative view of the bi-cluster phase observed in the antiferromagnetic system, as dynamical order allowed by the rotating solution of the FPE. This rotating solution is found to be neutrally stable down to zero temperature. Furthermore, the rotating solution can give rise to any degree of clustering, if the initial condition is appropriately chosen, in addition to bi-clustering. It would thus be of interest to probe such multi-cluster motions as tri-clustering, etc, by means of numerical simulations.

This paper is organized as follows: in section 2 we describe how the system of globally coupled rotors can be treated in a unified framework from the set of Langevin equations and the corresponding FPE. Various solutions of the FPE are given in section 3. It is shown that multi-cluster solutions emerge, manifesting dynamical order for the non-stationary rotating solution of the FPE. Section 4 is devoted to the stability analysis of the stationary solutions, with emphasis on the ferromagnetically coherent phase (with single cluster motion or thermodynamic order). The stability analysis of the non-stationary solution is presented in section 5, with a special focus on the antiferromagnetic case. Finally, a brief summary is given in section 6.

2. System of coupled rotors

We consider a system of N classical rotors, each of which is described by its phase angle and coupled sinusoidally to others. The dynamics of the coupled rotor system is governed by the set of equations of motion for the phase ϕ_i ($i = 1, \ldots, N$) of the ith rotor:

$$M\ddot{\phi}_i + \sum_j J_{ij}\sin(\phi_i - \phi_j) = 0, \tag{1}$$

where M is the inertia of each rotor and J_{ij} represents the coupling strength between rotors i and j. With the introduction of the canonical momentum $p_i = M\dot{\phi}_i$, the above equations are transformed into a set of canonical equations:

$$\dot{\phi}_i = \frac{\partial \mathcal{H}_N}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}_N}{\partial \phi_i} \tag{2}$$

with the N-particle Hamiltonian

$$\mathcal{H}_N = \sum_i \frac{p_i^2}{2M} - \sum_{i<j} J_{ij}\cos(\phi_i - \phi_j), \tag{3}$$

on which the microcanonical description is based.

On the other hand, in the canonical description the system is in contact with a heat reservoir of temperature T and described, in a most general way, by the set of Langevin equations:

$$M\ddot{\phi}_i + \Gamma\dot{\phi}_i + \sum_j J_{ij}\sin(\phi_i - \phi_j) = \eta_i, \tag{4}$$

where Γ is the damping coefficient and the Gaussian white noise $\eta_i(t)$ is characterized by the average $\langle \eta_i(t)\rangle = 0$ and the correlation $\langle \eta_i(t)\eta_j(t')\rangle = 2\Gamma T\delta_{ij}\delta(t - t')$. To derive the corresponding FPE, we write the equations of motion in the form

$$\dot{\phi}_i = \frac{p_i}{M} \qquad \dot{p}_i = -\frac{\Gamma}{M}p_i - \sum_j J_{ij}\sin(\phi_i - \phi_j) + \eta_i. \tag{5}$$

It is then straightforward to derive, via the standard procedure [14], the FPE for the probability distribution $P(\phi_i, p_i, t)$:

$$\frac{\partial P}{\partial t} = -\sum_i \frac{p_i}{M}\frac{\partial P}{\partial \phi_i} + \sum_i \frac{\partial}{\partial p_i}\left[\frac{\Gamma}{M}p_i + \sum_j J_{ij}\sin(\phi_i - \phi_j) + \Gamma T\frac{\partial}{\partial p_i}\right]P. \tag{6}$$

One may also derive the FPE for the Hamiltonian dynamics, which just reads equation (6) with $\Gamma = 0$. While reflecting that equation (4) with Γ set equal to zero reduces to equation (1), this suggests that equation (6) should provide the starting point for both descriptions: the microcanonical one ($\Gamma = 0$) and the canonical one ($\Gamma \neq 0$). In particular, the stationary solution of equation (6) is given by the canonical distribution $P^{(0)}(\phi_i, p_i) \propto \mathrm{e}^{-\mathcal{H}_N/T}$, describing equilibrium, with the very Hamiltonian in equation (3) *regardless of* Γ being zero or not. Note, however, that unlike the canonical ensemble where T represents the given temperature, in the microcanonical ensemble T still remains as an arbitrary parameter. In the latter, one may adjust T to the average kinetic energy, which allows the interpretation of T as the temperature. This prescription thus establishes correspondence between the two ensembles. Note also that in the zero-temperature limit ($T \to 0$), equation (4) reduces to the Caldirola–Kanai Hamiltonian dynamics [15], which needs external driving to have a nontrivial stationary state.

In order to measure a variety of coherence in the system, we conveniently introduce the generalized order parameter $\Delta^{(\ell)}$ defined by

$$\frac{1}{N}\sum_{i}^{N} \mathrm{e}^{\mathrm{i}\ell\phi_i} \equiv \Delta^{(\ell)}\, \mathrm{e}^{\mathrm{i}\theta_\ell}. \tag{7}$$

Apart from the global phase θ_ℓ, non-vanishing values of the order parameter $\Delta^{(\ell)}$ imply that rotors move as clusters, since rotors separated with phase angle $2\pi/\ell$ make contributions to $\Delta^{(\ell)}$. It thus can be used as a measure of the distribution of rotors, particularly, the degree of *clustering*. For instance, a non-vanishing value for $\ell = 1$ corresponds to the emergence of a mono-cluster (or magnetization), that for $\ell = 2$ corresponds to bi-cluster formation (with separation of π), and so on. Note that the $\ell = 2$ case may be regarded as the analogue of staggered magnetization in the short-range model.

3. Stationary and non-stationary solutions

In the infinite-range limit ($J_{ij} = J/N$ with $\to \infty$), we use equation (7) for $\ell = 1$,

$$\Delta^{(1)} = \frac{1}{N}\sum_{i} \mathrm{e}^{\mathrm{i}(\phi_i - \theta_1)}, \tag{8}$$

and decouple the set of equations of motion into a single-particle equation

$$M\ddot{\phi}_i + \Gamma\dot{\phi}_i + J\Delta^{(1)}\sin(\phi_i - \theta_1) = \eta_i, \tag{9}$$

satisfied by all rotors. Henceforth we therefore drop the rotor index i in equation (9), which leads to the standard FPE for the single-rotor probability distribution $P(\phi, p, t)$:

$$\frac{\partial P}{\partial t} = -\frac{p}{M}\frac{\partial P}{\partial \phi} + J\Delta^{(1)}\sin(\phi - \theta_1)\frac{\partial P}{\partial p} + \Gamma\frac{\partial}{\partial p}\left[\frac{p}{M} + T\frac{\partial}{\partial p}\right]P. \tag{10}$$

In the absence of damping ($\Gamma = 0$), this reduces to the FPE for the microcanonical ensemble:

$$\frac{\partial P}{\partial t} = -\frac{p}{M}\frac{\partial P}{\partial \phi} + J\Delta^{(1)}\sin(\phi - \theta_1)\frac{\partial P}{\partial p}, \tag{11}$$

which is also referred to as the Vlasov equation in some of the literature [3, 11, 12]. In terms of this probability distribution, the generalized order parameter is defined to be

$$\Delta^{(\ell)}\,\mathrm{e}^{\mathrm{i}\theta_\ell} = \langle \mathrm{e}^{\mathrm{i}\ell\phi}\rangle = \int \mathrm{d}p\,\mathrm{d}\phi\,\mathrm{e}^{\mathrm{i}\ell\phi}P(\phi, p, t). \tag{12}$$

3.1. Stationary solutions

For the sake of completeness, we briefly review the results for stationary solutions of the FPE [13]. As pointed out for the general case, both equations (10) and (11) support the *same* stationary ($\partial P/\partial t = 0$) solution:

$$P^{(0)}(\phi, p) = \frac{1}{\mathcal{Z}}\,\mathrm{e}^{-\mathcal{H}/T} \tag{13}$$

with the single-particle Hamiltonian

$$\mathcal{H} = \frac{p^2}{2M} - J\Delta^{(1)}\cos(\phi - \theta_1), \tag{14}$$

where the overall phase θ_1 manifests the global $U(1)$ symmetry. It is thus expected that both ensembles exhibit the same equilibrium behaviour.

One, however, should recall again that here T is given in equation (10) (for the canonical ensemble) but remains arbitrary in equation (11) (for the microcanonical ensemble). In the microcanonical ensemble the temperature should be defined as a measure of the average kinetic energy according to $\langle p^2\rangle/2M \equiv T/2$. The partition function is determined by normalization:

$$\mathcal{Z} = \int \mathrm{d}p \int \frac{\mathrm{d}\phi}{2\pi}\,\mathrm{e}^{-\mathcal{H}/T}. \tag{15}$$

For later use, we first describe some equilibrium properties of the globally coupled rotors [3] through the use of the single-particle model. Defining $x \equiv J\Delta^{(1)}/T$ and making use of the expansion

$$\mathrm{e}^{x\cos(\phi-\theta_1)} = \sum_{n=-\infty}^{\infty} I_n(x)\,\mathrm{e}^{\mathrm{i}n(\phi-\theta_1)} \tag{16}$$

with $I_n(x)$ being the modified Bessel function of the nth order, we evaluate the partition function as

$$\mathcal{Z} = \sqrt{2\pi MT}\,I_0(x). \tag{17}$$

We emphasize again that this approach based on the FPE provides a unified description of microcanonical and canonical ensembles and both ensembles generate the same equilibrium behaviour, determined by the same distribution $P^{(0)}(\phi, p)$. Namely, in both ensembles the generalized order parameter in equilibrium is given by

$$\Delta^{(\ell)}\,\mathrm{e}^{\mathrm{i}\theta_\ell} = \langle \mathrm{e}^{\mathrm{i}\ell\phi}\rangle = \int \mathrm{d}p \int \frac{\mathrm{d}\phi}{2\pi} P^{(0)}(\phi, p)\,\mathrm{e}^{\mathrm{i}\ell\phi}. \tag{18}$$

With the expansion in equation (16) and integration over ϕ, the order parameter reads[4]

$$\Delta^{(\ell)} = \frac{I_\ell(x)}{I_0(x)}. \tag{19}$$

Note here that $\Delta^{(\ell)}$ has an explicit dependence on the coherence order parameter $\Delta^{(1)}$ through $x \equiv J\Delta^{(1)}/T$. For $\ell = 1$, describing the emergence of coherence (the mono-cluster as thermodynamic order), equation (18) becomes an equation to be solved self-consistently:

$$\frac{T}{J}x = \frac{I_1(x)}{I_0(x)}. \tag{20}$$

This self-consistency equation determines whether the system exhibits coherence: the ordered phase ($\Delta^{(1)} \neq 0$) emerges when T/J is smaller than the slope of $I_1(x)/I_0(x)$ at $x = 0$, which is $1/2$. Accordingly, the ferromagnetic system ($J > 0$) undergoes a phase transition at the critical temperature $T_\mathrm{c} = J/2$. In the case of antiferromagnetic coupling ($J < 0$), on the other hand, equation (20) becomes $-Tx/|J| = I_1(x)/I_0(x)$, leading to the only solution $x = 0$. It is thus concluded that the antiferromagnetic system has no phase transition at finite temperatures (no mono-cluster). It is obvious in equation (19) that $\Delta^{(\ell)}$ for higher values of ℓ can assume nonzero values only for $\Delta^{(1)} \neq 0$; this implies that only the ferromagnetic system can develop

[4] In fact, there is a phase difference of $\theta_\ell - \theta_1$ for arbitrary ℓ, which may be disregarded. For the incoherent phase the global phase θ_1 does not come in.

all degrees of clustering below T_c. This is not surprising since the mono-cluster phase has a 2π symmetry and therefore is invariant under any rotations of multiples of 2π, which in turn gives rise to nonzero $\Delta^{(\ell)}$.

For $\Delta^{(1)} = 0$, describing the incoherent phase, the single-particle Hamiltonian (14) has only the kinetic energy term, thus reducing the canonical distribution $P^{(0)}(\phi, p)$ to the Maxwell distribution for both ensembles. Unlike equation (10), however, equation (11), the FPE in the microcanonical ensemble, allows an extra solution of the form $P^{(0)}(\phi, p) = f_0(p)$, an arbitrary function of p without ϕ-dependence, including the Maxwell distribution as a special case [13]. The only constraint is the normalization, and the distribution $P^{(0)}(\phi, p)$ uniform in ϕ guarantees $\Delta^{(1)} = 0$. As a result, $\Delta^{(\ell)}$ vanishes for all values of ℓ as well, and no multi-clustering is allowed by this type of stationary solution present only in the microcanonical ensemble.

3.2. Rotating solutions

In addition to the stationary solutions presented above, the FPE in the microcanonical ensemble also carries non-stationary solutions which have some significance for the antiferromagnetic system. For $\Delta^{(1)} = 0$, equation (11) becomes

$$\frac{\partial P}{\partial t} = -\frac{p}{M}\frac{\partial P}{\partial \phi}, \tag{21}$$

which has a solution of the general form $P^{(0)}(\phi, p, t) = u(\phi - pt/M, p)$. This is a rotating solution in the sense that the phase grows continuously with time with a continuous frequency spectrum ($\omega \propto p/M$). Requiring periodicity in ϕ, we write

$$P^{(0)}(\phi, p, t) = \sum_k \mathrm{e}^{\mathrm{i}k(\phi - \frac{p}{M}t)} F_k(p), \tag{22}$$

where $F_k(p)$ is an arbitrary function of p satisfying $F_{\pm 1}(p) = 0$ due to the condition $\Delta^{(1)} = 0$. The generalized order parameter for this solution is computed according to

$$\begin{aligned}\Delta^{(\ell)}\,\mathrm{e}^{\mathrm{i}\theta_\ell} &= \int \mathrm{d}p \int \frac{\mathrm{d}\phi}{2\pi} \sum_k \mathrm{e}^{\mathrm{i}\ell\phi}\,\mathrm{e}^{\mathrm{i}k(\phi - \frac{p}{M}t)} F_k(p) \\ &= \int \mathrm{d}p\,\mathrm{e}^{+\mathrm{i}\ell\frac{p}{M}t} F_{-\ell}(p),\end{aligned} \tag{23}$$

which shows that the higher order moment $\Delta^{(\ell)}$ in general does not vanish unless $F_{-\ell}(p) = 0$. Thus far there is no difference between the ferromagnetic and the antiferromagnetic couplings. As will be shown later, however, the stability of the rotating solution differs substantially, depending on the nature of the interaction. The rotating solution exists only for $\Delta^{(1)} = 0$, regardless of whether the system is in equilibrium or not. While such an incoherent phase appears only at high temperatures in the ferromagnetic case, $\Delta^{(1)}$ always remains zero at all temperature ranges in the antiferromagnetic case. Moreover, the rotation frequency gets higher as the order of the moment increases. This suggests that at low temperatures where thermal fluctuations are small, the phases with non-vanishing higher moments (high degrees of clustering) are easier to observe in the antiferromagnetic system than in the ferromagnetic one. In fact, this is precisely what has been seen in recent numerical simulations, which reported the bi-cluster phase in the antiferromagnetic system at very low temperatures [12]. The bi-cluster state, with two clusters separated by angle π, may be obtained with suitable choices of $F_k(p)$

in equation (22). For example, with the choice $F_{2k}(p) = F(p)$ and $F_{2k+1}(p) = 0$, we obtain[5]

$$P^{(0)}(\phi, p, t) = \pi F(p)\left[\delta\left(\phi - \frac{p}{M}t\right) + \delta\left(\phi - \frac{p}{M}t + \pi\right)\right]. \tag{24}$$

For another choice, say $F_{2k}(p) = (-1)^k F(p)$ and $F_{2k+1}(p) = 0$, one obtains

$$P^{(0)}(\phi, p, t) = \pi F(p)\left[\delta\left(\phi - \frac{p}{M}t + \frac{\pi}{2}\right) + \delta\left(\phi - \frac{p}{M}t - \frac{\pi}{2}\right)\right]. \tag{25}$$

All these are shown to be neutrally stable in section 5.

Equation (23) further indicates that there can exist higher order multi-cluster phases (for $\ell = 2, 3, \ldots$) as well, if appropriate choices for $F_k(p)$ are made. Recall again that the multi-cluster phase does not occur for stationary solutions since $\Delta^{(\ell)} = 0$ for time-independent solutions. In other words, the multi-cluster must rotate with higher frequency as the number of clusters grows; this suggests that the multi-cluster with large ℓ should be difficult to observe.

4. Stability of stationary states

In the previous work [13], we have already shown that the stability of the incoherent phase depends on the solutions of the FPE, providing a plausible explanation as to the physical origin of the quasi-stationarity. We now extend the analysis further to include the case of the coherent phase. For this purpose, we write the FPE, setting $\Delta^{(1)} \equiv \Delta$ and $\theta_1 \equiv \theta$, in the form

$$\frac{\partial P}{\partial t} = -\frac{p}{M}\frac{\partial P}{\partial \phi} + J\Delta\sin(\phi - \theta)\frac{\partial P}{\partial p} + \Gamma\frac{\partial}{\partial p}\left[\frac{p}{M} + T\frac{\partial}{\partial p}\right]P. \tag{26}$$

To probe the stability, we add a small perturbation to write

$$P(\phi, p, t) = P_0(\phi, p, t) + f(\phi, p, t) \tag{27}$$

and accordingly

$$\begin{aligned}\Delta(t) &= \Delta_0(t) + \Delta_1(t)\\ &= \int \mathrm{d}p \int \frac{\mathrm{d}\phi}{2\pi}\, \mathrm{e}^{\mathrm{i}(\phi-\theta)}\, [P_0(\phi, p, t) + f(\phi, p, t)]\,.\end{aligned} \tag{28}$$

Substituting these into (26), one obtains, to the lowest order,

$$\frac{\partial f}{\partial t} = -\frac{p}{M}\frac{\partial f}{\partial \phi} + J\Delta_1\sin(\phi - \theta)\frac{\partial P_0}{\partial p} + J\Delta_0\sin(\phi - \theta)\frac{\partial f}{\partial p} + \Gamma\frac{\partial}{\partial p}\left(\frac{p}{M} + T\frac{\partial}{\partial p}\right)f. \tag{29}$$

Since $f(\phi, p, t)$ and $\Delta_1(t)$ are periodic in ϕ, one can Fourier expand them in plane waves:

$$f(\phi, p, t) = \sum_k \int \mathrm{d}\omega\, \mathrm{e}^{\mathrm{i}(k\phi - \omega t)}\, \tilde{f}_k(p, \omega) \tag{30}$$

and

$$\begin{aligned}\Delta_1(t) &= \int \mathrm{d}p \frac{\mathrm{d}\phi}{2\pi}\, \mathrm{e}^{\mathrm{i}(\phi-\theta)} f(\phi, p, t)\\ &= \int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t} \int \mathrm{d}p\, \tilde{f}_{-1}(p, \omega),\end{aligned} \tag{31}$$

where the integration over ϕ has been performed. Note here that the perturbed order parameter is proportional only to $\tilde{f}_{-1}(p, \omega)$ (or to $\tilde{f}_{+1}(p, \omega)$ if the order parameter has been defined

[5] Here the summation of the infinite series in equation (22) yields $P^{(0)}(\phi, p, t) = 2\pi F(p)\delta(2(\phi - pt/M))$ with 2π-periodicity in the argument. From this equation, manifesting the periodicity, we obtain equation (24). Similarly, equation (25) is obtained.

to be $\Delta = \langle e^{-i(\phi-\theta)} \rangle$). Inserting these expressions into equation (29) and collecting coefficients of $e^{i(k\phi-\omega t)}$, one finds the relations satisfied by the Fourier coefficients $\tilde{f}_k(p,\omega)$.

In the case of ferromagnetic coupling, the coherent phase ($\Delta_0 \neq 0$) arises at temperatures below T_c, regardless of the presence of damping. The stationary solution in equation (13) can be written, with the help of equations (16), (17) and (19), in the form

$$\begin{aligned} P_0(\phi, p) &= f_M(p) \sum_{n=-\infty}^{\infty} \frac{I_n(x)}{I_0(x)} e^{in(\phi-\theta)} \\ &= f_M(p) \sum_{n=-\infty}^{\infty} \Delta^{(n)}(x)\, e^{in(\phi-\theta)}, \end{aligned} \tag{32}$$

where $f_M(p) \equiv (2\pi MT)^{-1/2} \exp(-p^2/2MT)$ is the Maxwell distribution and $x \equiv J\Delta_0/T$ as before. When $x = 0$, the above equation simply reduces to the Maxwell distribution, which is stable at temperatures above T_c. Our concern now is how the coherent phase gets its stability as the temperature is lowered below the critical temperature. Putting equations (31) and (32) into equation (29), one obtains the following equation for the Fourier coefficients $\tilde{f}_k(p,\omega)$:

$$\begin{aligned} &\left(\omega - \frac{kp}{M}\right) \tilde{f}_k - \frac{J\Delta_0}{2} \frac{\partial}{\partial p} (\tilde{f}_{k-1} - \tilde{f}_{k+1}) - i\Gamma \frac{\partial}{\partial p} \left(\frac{p}{M} + T \frac{\partial}{\partial p} \right) \tilde{f}_k \\ &\qquad = \frac{J}{2} [\Delta^{(k-1)}(x) - \Delta^{(k+1)}(x)] f'_M(p) \int dp' \tilde{f}_{-1}. \end{aligned} \tag{33}$$

We note here that the emergence of coherence contributes to the off-diagonal term in equation (33) and to the appearance of higher order generalized order parameters, making the stability analysis non-trivial. When $\omega - kp/M = 0$, we have a continuous spectrum, and for $\Gamma = 0$, equation (33) becomes

$$\frac{J\Delta_0}{2} \frac{\partial}{\partial p} \tilde{f}_{k-1} = \frac{J}{2} \Delta^{(k-1)}(x) f'_M(p) \left[\int dp' \tilde{f}_{-1} \right]. \tag{34}$$

It is easy to show by direct substitution that this equation has a solution of the form

$$\tilde{f}_k(p,\omega) = \begin{cases} f_M(p) h_k(\omega), & \text{for } k \neq \pm 1, 0, -2 \\ 0, & \text{otherwise.} \end{cases} \tag{35}$$

It is of interest to note this is also the solution for $\Gamma \neq 0$ as well, since the term including Γ vanishes for the Maxwell distribution. For $\omega - kp/M \neq 0$, we have a discrete spectrum and may not solve the equation for the general case. Still we may proceed further if we take the phase-only perturbation, namely, $f(p,\phi,t) = f_M(p)h(\phi,t)$. We then have the Fourier coefficient $\tilde{f}_k(p,\omega) = f_M(p)h_k(\omega)$ with $h_k(\omega)$ being the Fourier coefficient of $h(\phi,t)$, which in turn gives $(\partial/\partial p)\tilde{f}_k(p,\omega) = f'_M(p)h_k(\omega)$ and $\int dp\, \tilde{f}_k(p,\omega) = h_k(\omega)$. Further, the term including Γ vanishes identically in this case. Dividing equation (33) by $\omega - kp/M$ and integrating over p, we obtain

$$h_k(\omega) = -[\Delta^{(k-1)}(x) - \Delta^{(k+1)}(x)]\chi_k(\omega)\tilde{h}_{-1}(\omega) + 2\Delta_0 \chi_k(\omega)[h_{k+1}(\omega) - h_{k-1}(\omega)], \tag{36}$$

where we have introduced the k-dependent response function

$$\chi_k(\omega) = \frac{J}{2} \int dp \frac{f'_M(p)}{\omega + kp/M} \tag{37}$$

and used equation (A.4). Some properties of this response function, which is frequency-dependent, are discussed separately in the appendix. For $x \neq 0$, the recursion relation for the modified Bessel functions [16]:

$$I_{k-1}(x) - I_{k+1}(x) = \frac{2k}{x} I_k(x) \tag{38}$$

leads equation (36) to take the form

$$h_k - 2\Delta_0\chi_k(\omega)(h_{k+1} - h_{k-1}) = \frac{2k}{x}\Delta^{(k)}(x)\chi_k(\omega)h_{-1}, \tag{39}$$

which needs to be solved. For $k = 0$, from equations (36) and (A.2), we find $h_0 = 0$, implying the absence of a constant term in the perturbation. Noting $\Delta^{(k)}(x) = \Delta^{(-k)}(x)$ and $\chi_k(\omega) = -\chi_{-k}(\omega)$, we write the difference equation in the matrix form,

$$\Lambda \begin{pmatrix} h_{-1} \\ h_{-2} \\ h_{-3} \\ \cdots \end{pmatrix} = 0 \tag{40}$$

with the matrix

$$\Lambda = \begin{pmatrix} 1 + (2/x)\Delta^{(1)}\chi_1 & -\Delta_0\chi_1 & 0 & \cdots \\ \Delta_0\chi_2 + (4/x)\Delta^{(2)}\chi_2 & 1 & -\Delta_0\chi_2 & \cdots \\ (6/x)\Delta^{(3)}\chi_3 & \Delta_0\chi_3 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \tag{41}$$

Here we have included the terms with only negative k values, reflecting that all order parameters are defined by equation (18). In order to have non-trivial solutions for $\vec{h} = (h_{-1}, h_{-2}, h_{-3}, \ldots)$, one should have the vanishing determinant:

$$\varepsilon(\omega) \equiv \det\Lambda = 0. \tag{42}$$

Let us first consider the limit $\Delta_0 \to 0$ or $x = J\Delta_0/T \to 0$, corresponding to the incoherent phase. In this limit all the off-diagonal terms vanish, since $I_0(x) \to 1$ and $I_n(x) \to (x/2)^n$ so that $\Delta^{(n)} \to (x/2)^n$. Equation (40) then becomes

$$\begin{pmatrix} 1+\chi_1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} h_{-1} \\ h_{-2} \\ h_{-3} \\ \cdots \end{pmatrix} = 0, \tag{43}$$

which leads to

$$1 + \chi_1(\omega) = 1 + \chi(\omega) \equiv 1 + \frac{JM}{2}\tilde{\chi}(\omega) = 0 \tag{44}$$

for non-vanishing h_{-1}, while all other h are zero. The detailed analytic properties of the reduced response function $\tilde{\chi}(\omega) \equiv (2/JM)\chi(\omega)$, with the complex frequency $\omega = \omega_r + \mathrm{i}\omega_i$, are presented in the appendix. Equation (44) describes the condition for the incoherent phase with the Maxwell distribution, which is stable/unstable above/below T_{c} [13]. For $x \neq 0$, in principle we have to solve equation (42) including all the terms in equation (41). Since this is very formidable, we instead consider just a few terms to explore how the stability of the solution changes. To this end, we write $\varepsilon(\omega) \approx \varepsilon^{(m)}(\omega)$, the determinant obtained when the first m Fourier components are kept. With only the first Fourier component h_{-1} considered, equation (42) obtains the form

$$\begin{aligned} \varepsilon^{(1)}(\omega) &= 1 + \frac{2}{x}\Delta^{(1)}\chi_1(\omega) \\ &= 1 + \frac{2T}{J}\chi(\omega) \\ &= 1 + TM\tilde{\chi}(\omega) = 0, \end{aligned} \tag{45}$$

for which equations (A.11) to (A.13) yield $\omega_i = 0$ as the only solution. Comparison of equation (45) with equation (44) shows the correspondence $T = J/2 = T_{\mathrm{c}}$; this indicates that the solution is neutrally stable at the critical temperature, below which coherence develops. Including the next component h_{-2}, one has

$$\varepsilon^{(2)}(\omega) = \varepsilon^{(1)}(\omega) + \Delta_0 \left(\Delta_0 + \frac{4}{x}\Delta^{(2)} \right) \chi_1(\omega)\chi_2(\omega) = 0, \tag{46}$$

which, with $\Delta^{(1)} = \Delta_0$ and $\chi_k(\omega) = \chi(\omega/k)/k$, becomes

$$\varepsilon^{(2)}(\omega) = 1 + T\tilde{\chi}(\omega) + \frac{T^2}{8}\left[x^2 + 4x\frac{I_2(x)}{I_1(x)} \right] \tilde{\chi}(\omega)\tilde{\chi}(\omega/2) = 0. \tag{47}$$

Since $\tilde{\chi}(\omega)$ and $\tilde{\chi}(\omega/2)$ have the same pole structure, the real and the imaginary parts of equation (47) read

$$\begin{aligned} \mathrm{Re}\,\varepsilon^{(2)}(\omega) \equiv{}& 1 + T\,\mathrm{Re}\,\tilde{\chi}(\omega) + \frac{T^2}{8}\left[x^2 + 4x\frac{I_2(x)}{I_1(x)} \right] \\ & \times [\mathrm{Re}\,\tilde{\chi}(\omega)\mathrm{Re}\,\tilde{\chi}(\omega/2) - \mathrm{Im}\,\tilde{\chi}(\omega)\,\mathrm{Im}\,\tilde{\chi}(\omega/2)] = 0 \end{aligned} \tag{48}$$

$$\begin{aligned} \mathrm{Im}\,\varepsilon^{(2)}(\omega) \equiv{}& T\,\mathrm{Im}\,\tilde{\chi}(\omega) + \frac{T^2}{8}\left[x^2 + 4x\frac{I_2(x)}{I_1(x)} \right] \\ & \times [\mathrm{Im}\tilde{\chi}(\omega)\,\mathrm{Re}\,\tilde{\chi}(\omega/2) + \mathrm{Re}\,\tilde{\chi}(\omega)\mathrm{Im}\,\tilde{\chi}(\omega/2)] = 0. \end{aligned} \tag{49}$$

In the appendix it is shown that $\omega_r = 0$ is a solution of $\mathrm{Im}\,\tilde{\chi}(\omega) = 0$, implying that this is also a solution of $\mathrm{Im}\,\varepsilon^{(2)}(\omega) = 0$. For $\omega_i > 0$, equation (48) becomes

$$f(y) - 1 = \frac{1}{8}\left[x^2 + 4x\frac{I_2(x)}{I_1(x)} \right] f(y)f(y/2) \tag{50}$$

with $y \equiv \omega_i\sqrt{M/2T}$. As y increases from zero to arbitrarily large values, the left-hand side of equation (50) decreases monotonically from zero to -1 while the right-hand side is positive-definite for $y > 0$. This suggests that there is no solution for $\omega_i > 0$ to make the system unstable. For $\omega_i = 0$, which corresponds to the neutral stability, we have $\mathrm{Re}\,\tilde{\chi}(\omega) = -1/T$ and thus equation (48) reads

$$x^2 + 4x\frac{I_2(x)}{I_1(x)} = 0, \tag{51}$$

which leads to $x = 0$ for the critical case. For $\omega_i = -|\omega_i| < 0$, for which the system becomes stable, equation (48) takes the form

$$g(y) - 1 = \frac{1}{8}\left[x^2 + 4x\frac{I_2(x)}{I_1(x)} \right] g(y)g(y/2). \tag{52}$$

As shown in the appendix, $g(y)$ is a monotonically increasing function of y from unity to infinity, and accordingly, $g(y)g(y/2)$ is also a monotonically increasing function of y in the same domain. Since the left-hand side of the above equation is monotonically increasing from zero to arbitrarily large values, equation (52) allows a solution only for some range of x values. We have determined numerically the range of x values, in which there exists a solution for $y > 0$, to find

$$x \equiv \frac{J\Delta_0}{T} < x_c^{(2)} \approx 1.32. \tag{53}$$

This indicates that the coherent solution is stable only at temperatures above T_0, at which $\Delta_0(T)$ and $x_c T/J$ meet. We see that the stable region does not extend to the zero temperature, presumably because we have included only the second component in our analysis (the first component is trivial). This may be resolved if one include higher components. Adding the third component h_{-3} leads to the following equation,

$$\varepsilon^{(3)}(\omega) = \varepsilon^{(2)}(\omega) + \frac{1}{24}J^2\Delta_0^2\left[1 + T\tilde{\chi}(\omega) + 3T\frac{I_3(x)}{I_1(x)}\tilde{\chi}(\omega)\right]\tilde{\chi}(\omega/2)\tilde{\chi}(\omega/3),$$
$$= 0 \tag{54}$$

from which one can perform a similar analysis to find

$$\left[1 + \frac{x^2}{24}f(y/2)f(y/3)\right][f(y) - 1] = \frac{1}{8}\left[x^2 + 4x\frac{I_2(x)}{I_1(x)} - \frac{I_3(x)}{I_1(x)}f(y/3)\right]f(y)f(y/2) \tag{55}$$

for $\omega_i > 0$ and

$$\left[1 + \frac{x^2}{24}g(y/2)g(y/3)\right][g(y) - 1] = \frac{1}{8}\left[x^2 + 4x\frac{I_2(x)}{I_1(x)} - \frac{I_3(x)}{I_1(x)}g(y/3)\right]g(y)g(y/2) \tag{56}$$

for $\omega_i < 0$. Again, equation (55) does not have a solution for positive y since the left-hand side is less than zero while the right-hand side is greater than zero. Equation (56) is found to have a solution for $x < x_c^{(3)} \approx 1.51$. Note here that x_c is increased substantially once the third component is included, implying that T_0, above which the coherent solution is stable, is lowered. We have performed this analysis, including up to four components, and confirmed that this trend persists; this suggests the plausible conjecture that the coherent solution is stable down to zero temperature if all the Fourier components are included.

We now turn our attention to the stability of the antiferromagnetic system for which there is no equilibrium order ($\Delta_0 = 0$ or $x = 0$). The stability equation reads, for $J = -|J|$,

$$1 - \frac{|J|}{2}\tilde{\chi}(\omega) = 0, \tag{57}$$

which, depending on the sign of ω_i (with $\omega_r = 0$), becomes

$$\begin{cases} 1 + (|J|/2T)f(y) = 0 & \text{for} \quad \omega_i > 0 \\ 1 + |J|/2T = 0 & \text{for} \quad \omega_i = 0 \\ 1 + (|J|/2T)g(y) = 0 & \text{for} \quad \omega_i < 0. \end{cases} \tag{58}$$

None of these equations has a solution, since $f(y)$ and $g(y)$ are positive-definite. This means that the antiferromagnetic system cannot have self-sustained deviation in the absence of the perturbation with a discrete spectrum. On the other hand, with the continuous spectrum $\omega = \omega_r = kp/M$, the system is neutrally stable at all temperatures.

5. Stability of non-stationary states

As mentioned in section 3, the non-stationary solution exists only in the microcanonical ensemble ($\Gamma = 0$) with a continuous frequency spectrum, which can develop spontaneously. Our concern in this section is the stability of this solution, especially in the case of the antiferromagnetic interaction. Equation (29) for stability reads, with $\Delta_0 = \Gamma = 0$,

$$\frac{\partial f}{\partial t} = -\frac{p}{M}\frac{\partial f}{\partial \phi} + J\Delta_1 \sin\phi \frac{\partial P_0}{\partial p}, \tag{59}$$

where P_0 is given by equation (22). The last term in the above equation obtains the form

$$J\Delta_1 \sin\phi \frac{\partial P_0}{\partial p} = \frac{J}{2\mathrm{i}}\int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t}\left[\int \mathrm{d}p'\, \tilde{f}_{-1}(p', \omega)\right](\mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{-\mathrm{i}\phi})$$
$$\times \sum_k \mathrm{e}^{\mathrm{i}k\phi}\frac{\partial}{\partial p}\left[\exp\left(-\mathrm{i}\frac{kp}{M}t\right)F_k(p)\right]$$

$$= \frac{J}{2\mathrm{i}}\sum_k \frac{\partial}{\partial p}\int \mathrm{d}\omega[(\mathrm{e}^{\mathrm{i}(k+1)\phi} - \mathrm{e}^{\mathrm{i}(k-1)\phi})]\exp\left[-\mathrm{i}\left(\omega + \frac{kp}{M}\right)t\right]$$
$$\times F_k(p)\int \mathrm{d}p'\,\tilde{f}_{-1}(p',\omega)$$
$$= \frac{J}{2\mathrm{i}}\sum_k \int \mathrm{d}\omega\,\mathrm{e}^{\mathrm{i}(k\phi-\omega t)}\frac{\partial}{\partial p}[\tilde{F}_{k-1}(p,\omega) - \tilde{F}_{k+1}(p,\omega)] \tag{60}$$

with

$$\tilde{F}_k(p,\omega) \equiv F_k(p)\int \mathrm{d}p'\,\tilde{f}_{-1}\left(p', \omega - \frac{kp}{M}\right), \tag{61}$$

which leads to the equation for the Fourier coefficients:

$$\left(\omega - \frac{kp}{M}\right)\tilde{f}_k(p,\omega) = \frac{J}{2}\frac{\partial}{\partial p}[\tilde{F}_{k-1}(p,\omega) - \tilde{F}_{k+1}(p,\omega)]. \tag{62}$$

Since we are dealing with the perturbation of the non-stationary state with a continuous spectrum, the frequency of the perturbation should satisfy $\omega - kp/M \neq 0$; otherwise, there would be no perturbation at all. This allows us to divide equation (62) by $\omega - kp/M$ and to integrate over p. For $k = -1$, we have

$$\int \mathrm{d}p'\,\tilde{f}_{-1}(p',\omega) = \frac{J}{2}\int \mathrm{d}p\left(\omega + \frac{p}{M}\right)^{-1}\frac{\partial}{\partial p}[\tilde{F}_{-2}(p,\omega) - \tilde{F}_0(p,\omega)], \tag{63}$$

while for $k \neq -1$, $\tilde{f}_k(p,\omega)$ is determined by $\tilde{f}_{-1}(p', \omega \pm kp/M)$ through equations (61) and (62). It is thus enough to have non-vanishing $\tilde{f}_{-1}(p',\omega)$. Now suppose that $\omega = \omega_0$ is a solution of equation (63), i.e., $\int \mathrm{d}p'\,\tilde{f}_{-1}(p',\omega) \neq 0$ for $\omega = \omega_0$. If we write $\int \mathrm{d}p'\,\tilde{f}_{-1}(p',\omega) = K\delta(\omega - \omega_0)$, then $\int \mathrm{d}p'\,\tilde{f}_{-1}\big(p', \omega + \frac{2p}{M}\big) = K\delta\big(\omega + \frac{2p}{M} - \omega_0\big)$. Integration over ω gives

$$1 + \frac{JM}{2}\int \mathrm{d}p\frac{F_0'(p)}{p + M\omega_0} = \frac{JM}{2}\int \mathrm{d}p\frac{F_{-2}(p)}{(p - M\omega_0)^2}$$
$$= \frac{JM}{2}\int \mathrm{d}p\frac{F_{-2}'(p)}{p - M\omega_0}, \tag{64}$$

where the last line is obtained by integration by parts. Hence the frequency of a self-sustained oscillation and accordingly the stability is, similarly to the stationary case (equation (44)), determined by

$$1 + \frac{JM}{2}\int \mathrm{d}p\left[\frac{F_0'(p)}{p + M\omega_0} - \frac{F_{-2}'(p)}{p - M\omega_0}\right] = 0. \tag{65}$$

The stability condition is thus entirely the same as that of the stationary case except that we now have two momentum distributions: From equations (A.8) and (A.9) with $M\omega_0 = \tilde{\omega}_r + \tilde{\omega}_i$, we have

$$\begin{cases} \dfrac{2}{JM} + \displaystyle\int_{-\infty}^{\infty} \mathrm{d}p\left[\frac{(p+\tilde{\omega}_r)F_0'(p)}{(p+\tilde{\omega}_r)^2 + \tilde{\omega}_i^2} - \frac{(p-\tilde{\omega}_r)F_{-2}'(p)}{(p-\tilde{\omega}_r)^2 + \tilde{\omega}_i^2}\right] = 0 \\ \displaystyle\int_{-\infty}^{\infty} \mathrm{d}p\frac{F_0'(p)}{(p+\tilde{\omega}_r)^2 + \tilde{\omega}_i^2} - \frac{F_{-2}'(p)}{(p-\tilde{\omega}_r)^2 + \tilde{\omega}_i^2} = 0, \end{cases} \tag{66}$$

for $\omega_i > 0$, for which the system is unstable as the perturbation grows in time. In the opposite case ($\omega_i < 0$), the perturbation dies out to make the system stable. The condition for this is given by

$$\begin{cases} \dfrac{2}{JM} + \displaystyle\int_{-\infty}^{\infty} \mathrm{d}p \left[\dfrac{(p+\tilde{\omega}_r)F_0'(p)}{(p+\tilde{\omega}_r)^2+\tilde{\omega}_i^2} - \dfrac{(p-\tilde{\omega}_r)F_{-2}'(p)}{(p-\tilde{\omega}_r)^2+\tilde{\omega}_i^2} \right] \\ \quad + 2\pi [\mathrm{Im}\, F_0'(-\tilde{\omega}) - \mathrm{Im}\, F_{-2}'(-\tilde{\omega})] = 0 \\ \tilde{\omega}_i \displaystyle\int_{-\infty}^{\infty} \mathrm{d}p \left[\dfrac{F_0'(p)}{(p+\tilde{\omega}_r)^2+\tilde{\omega}_i^2} - \dfrac{F_{-2}'(p)}{(p-\tilde{\omega}_r)^2+\tilde{\omega}_i^2} \right] \\ \quad + 2\pi [\mathrm{Re}\, F_0'(-\tilde{\omega}) - \mathrm{Re}\, F_{-2}'(-\tilde{\omega})] = 0. \end{cases} \tag{67}$$

Finally, in the neutral case ($\omega_i = 0$), the condition simply reads

$$\begin{cases} \dfrac{2}{JM} + \mathcal{P} \displaystyle\int_{-\infty}^{\infty} \mathrm{d}p \left[\dfrac{F_0'(p)}{p+\tilde{\omega}_r} - \dfrac{F_{-2}'(p)}{p-\tilde{\omega}_r} \right] = 0 \\ F_0'(-\tilde{\omega}_r) - F_{-2}'(-\tilde{\omega}_r) = 0, \end{cases} \tag{68}$$

where $\mathcal{P}$ stands for the principal part. Our next task is to determine stability for specific distributions of $F_0(p)$ and $F_{-2}(p)$. Since most dynamical calculations, for both ferromagnetic and antiferromagnetic systems, have used the so-called water-bag distribution, we also consider the momenta to be distributed uniformly in the range $[-\alpha, \alpha]$:

$$F_0(p) = \pm F_{-2}(p) = \frac{1}{2\alpha}. \tag{69}$$

Substitution of $F_0(p) = \pm F_{-2}(p) = (2\alpha)^{-1}[\delta(p+\alpha) - \delta(p-\alpha)]$ into equations (66)–(68), depending on the sign of ω_i, determines the frequency ω_0.

We first consider the case $F_0(p) = F_{-2}(p)$. From the second equations of (66) (for $\omega_i > 0$) and (67) (for $\omega_i < 0$), we find $\omega_r = 0$, while there is no solution to satisfy the first ones. When $\omega_i = 0$, again there is no solution to satisfy the first equation of (68). This indicates that there is no self-sustained oscillation in the system. Note, however, that the system is neutrally stable as it has a continuous spectrum ($\omega = kp/M$). Next, when $F_0(p) = -F_{-2}(p)$, one finds

$$\omega_i = \pm\sqrt{\frac{J}{M} - \left(\frac{\alpha}{M}\right)^2}, \qquad \omega_r = 0 \qquad \text{for} \quad \alpha < \alpha_R \tag{70}$$

$$\omega_r = \pm\sqrt{-\frac{J}{M} + \left(\frac{\alpha}{M}\right)^2}, \qquad \omega_i = 0 \qquad \text{for} \quad \alpha > \alpha_R \tag{71}$$

with $\alpha_R \equiv \sqrt{JM}$. In the microcanonical ensemble one may relate the average kinetic energy to the temperature: $T/2 = \langle p^2 \rangle / 2M = \alpha^2/6M$, from which one has $T_R = \alpha_R^2 \big/ 3M = J/3$.[6] Accordingly, it is concluded in this case that the rotating solution is neutrally stable for $T > T_R$ and becomes unstable below T_R. Note here that T_R is lower than the equilibrium critical temperature $T_c = J/2$.

For the antiferromagnetic system ($J < 0$), we replace $J = -|J|$ in equation (70) to obtain, for $\omega_i = 0$,

$$\omega_r = \pm\sqrt{\frac{|J|}{M} + \left(\frac{\alpha}{M}\right)^2}, \tag{72}$$

while there is no solution for $\omega_r = 0$. We therefore conclude that the antiferromagnetic system is neutrally stable for all α, i.e., at all temperatures. In section 3 we have shown

[6] Numerical experiments usually use the internal energy U as the control parameter, which is related to the temperature T via $U = T/2 + (1 - \Delta^2)/2$ for $J = 1$. We here use a slightly different definition of the potential energy.

that the bi-cluster state is allowed by the rotating distribution. The result obtained here that this non-stationary solution is neutrally stable at all temperatures thus suggests an alternative explanation as to the origin of the spontaneously formed bi-cluster state in numerical simulations, which retains its form for quite a long time [12]. This is parallel with the emergence of quasi-stationarity in the ferromagnetic system, associated with the neutral stability [13].

6. Conclusion

In this paper, we have presented a detailed analysis of a system of globally coupled rotors. Starting from a set of Langevin equations and their corresponding Fokker–Planck equation, which includes the microcanonical ensemble approach as a limiting case, we have found a class of solutions and studied their stability. The standard canonical distribution constitutes a simultaneous solution of the canonical and the microcanonical ensembles, and thus describes the same equilibrium behaviour in both ensembles, leading to the coherent phase (characterized by a nonzero mono-cluster order parameter, i.e., $\Delta^{(1)} \neq 0$) below the critical temperature T_c in the ferromagnetic system. The stability of the coherent phase is governed by an infinite-order difference equation, the behaviour of which may be understood by considering successively higher order terms (i.e., Fourier components in the perturbation). It has been found that the coherent phase is stable above some temperature T_0, which is finite if one includes only a few of the lowest Fourier components. As more components are considered, T_0 tends to decrease towards zero; this leads us to surmise that an infinite number of Fourier components would stabilize the coherent phase down to zero temperature. Namely, it is expected that the stability equation, if treated exactly, leads to the stability of the coherent phase at all temperatures below T_c.

We find a more interesting possibility for the non-stationary (rotating) solution with regard to dynamical order. Dynamical order, manifested by multi-cluster motion, is allowed for both ferromagnetic and antiferromagnetic interactions. Unlike a ferromagnetic system, in which dynamical order ceases to exist below the temperature T_R, dynamical order is observed to be neutrally stable down to zero temperature in the antiferromagnetic system. This suggests an alternative explanation as to the origin of the spontaneous formation of the bi-cluster phase in the system of antiferromagnetically coupled rotors. This is in parallel with the explanation that the quasi-stationarity observed in ferromagnetically coupled rotors is related to the neutral stability of the stationary solution in the incoherent phase below the equilibrium critical temperature [13].

To conclude, we have introduced a unified approach for both the canonical ensemble and the microcanonical ensemble, based on the Fokker–Planck equation. Depending on the ensemble, the Fokker–Planck equation admits a few solutions which have implications for some of the remarkable features (quasi-stationarity in ferromagnetic systems and bi-cluster motion in antiferromagnetic systems) observed in numerical experiments. We provide natural explanations for the origin of those seemingly unrelated features within the same context. Finally, we point out that our approach is based on an effective one-particle dynamics, exact for an infinite number of particles and does not reflect instabilities that may be caused by the finiteness of the number of particles.

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Appendix. Properties of $\chi_k(\omega)$

In this appendix we describe some properties of the response function $\chi_k(\omega)$ for the Maxwell distribution $f_M(p)$:

$$\chi_k(\omega) = \frac{J}{2}\int \mathrm{d}p \frac{f_M'(p)}{\omega + kp/M}. \tag{A.1}$$

First, for $k = 0$, we have

$$\chi_0(\omega) = \frac{J}{2}\int \mathrm{d}p \frac{f_M'(p)}{\omega} = 0 \tag{A.2}$$

since $f_M'(p)$ is an odd function. We next write $k \to -k$ and change the integration variable p to $-p$ in equation (A.1) to get

$$\begin{aligned}\chi_{-k}(\omega) &= \frac{J}{2}\int \mathrm{d}p \frac{f_M'(p)}{\omega - kp/M} \\ &= \frac{J}{2}\int \mathrm{d}(-p) \frac{f_M'(-p)}{\omega + kp/M} \\ &= -\frac{J}{2}\int \mathrm{d}p \frac{f_M'(p)}{\omega + kp/M} = -\chi_k(\omega),\end{aligned} \tag{A.3}$$

again noting that $f_M'(p)$ is an odd function. Similarly, it is straightforward to show that

$$\chi_k(-\omega) = -\chi_{-k}(\omega) = \chi_k(\omega). \tag{A.4}$$

Further, equation (A.1) can also be written as

$$\begin{aligned}\chi_k(\omega) &= \frac{JM}{2k}\int \mathrm{d}p \frac{f_M'(p)}{p + M\omega/k} \\ &\equiv \frac{1}{k}\chi(\omega/k),\end{aligned} \tag{A.5}$$

where

$$\chi(\omega) \equiv \frac{JM}{2}\int \mathrm{d}p \frac{f_M'(p)}{p + M\omega} = \chi(-\omega) \tag{A.6}$$

is the response function already defined in section 4. Although we consider here the Maxwell distribution, the properties given above hold for any momentum distribution $f_0(p)$, only if it is an even function of p. We now proceed to evaluate this function, paying attention to the simple pole at $p = -M\omega$ on the complex p-plane. Setting $M\omega \equiv \tilde{\omega}$ and making analytic continuation $\omega = \omega_r + \mathrm{i}\omega_i$, we obtain $\chi(\omega)$ in the form

$$\tilde{\chi}(\omega) \equiv \frac{2}{JM}\chi(\omega) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \mathrm{d}p \frac{f_M'(p)}{p + \tilde{\omega}} & \text{for} \quad \omega_i > 0 \\ \displaystyle\mathcal{P}\int_{-\infty}^{\infty} \mathrm{d}p \frac{f_M'(p)}{p + \tilde{\omega}} - \mathrm{i}\pi f_M'(-\tilde{\omega}) & \text{for} \quad \omega_i = 0 \\ \displaystyle\int_{-\infty}^{\infty} \mathrm{d}p \frac{f_M'(p)}{p + \tilde{\omega}} - 2\mathrm{i}\pi f_M'(-\tilde{\omega}) & \text{for} \quad \omega_i < 0. \end{cases} \tag{A.7}$$

With the tilde sign omitted for convenience, the real part reads

$$\mathrm{Re}\,\tilde{\chi}(\omega) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \mathrm{d}p \frac{(p+\omega_r) f_M'(p)}{(p+\omega_r)^2 + \omega_i^2} & \text{for} \quad \omega_i > 0 \\ \displaystyle P\int_{-\infty}^{\infty} \mathrm{d}p \frac{f_M'(p)}{p+\omega_r} + \pi\,\mathrm{Im}\, f_M'(-\omega) & \text{for} \quad \omega_i = 0 \\ \displaystyle\int_{-\infty}^{\infty} \mathrm{d}p \frac{(p+\omega_r) f_M'(p)}{(p+\omega_r)^2 + \omega_i^2} + 2\pi\,\mathrm{Im}\, f_M'(-\omega) & \text{for} \quad \omega_i < 0 \end{cases} \tag{A.8}$$

while the imaginary part is given by

$$\mathrm{Im}\,\tilde{\chi}(\omega) = \begin{cases} \displaystyle\omega_i \int_{-\infty}^{\infty} \mathrm{d}p \frac{f_M'(p)}{(p+\omega_r)^2 + \omega_i^2} & \text{for} \quad \omega_i > 0 \\ \mathrm{Re}\, f_M'(-\omega) & \text{for} \quad \omega_i = 0 \\ \displaystyle\omega_i \int_{-\infty}^{\infty} \mathrm{d}p \frac{f_M'(p)}{(p+\omega_r)^2 + \omega_i^2} + 2\pi\mathrm{Re}\, f_M'(-\omega) & \text{for} \quad \omega_i < 0. \end{cases} \tag{A.9}$$

We next write

$$\begin{aligned} f_M'(\omega_r + \mathrm{i}\omega_i) &= \frac{\omega_r + \mathrm{i}\omega_i}{\sqrt{2\pi M^3 T^3}} \mathrm{e}^{-(\omega_r^2 - \omega_i^2)/2MT} \mathrm{e}^{-\mathrm{i}\omega_r\omega_i/MT} \\ &= -\frac{\mathrm{e}^{-(\omega_r^2-\omega_i^2)/2MT}}{\sqrt{2\pi M^3 T^3}} \left[\omega_r \cos\frac{\omega_r\omega_i}{MT} + \omega_i \sin\frac{\omega_r\omega_i}{MT} + \mathrm{i}\left(\omega_i \cos\frac{\omega_r\omega_i}{MT} - \omega_r \sin\frac{\omega_r\omega_i}{MT}\right)\right] \\ &\equiv \mathrm{Re}\, f_M'(\omega_r + \mathrm{i}\omega_i) + \mathrm{i}\,\mathrm{Im}\, f_M'(\omega_r + \mathrm{i}\omega_i), \end{aligned} \tag{A.10}$$

from which it is obvious that $\mathrm{Re}\, f_M'(\omega_r + \mathrm{i}\omega_i) = 0$ for $\omega_r = 0$ and $\mathrm{Im}\, f_M'(\mathrm{i}\omega_i) = -(2\pi M^3 T^3)^{-1/2}\omega_i\, \mathrm{e}^{\omega_i^2/2MT}$. We thus conclude that $\omega_r = 0$ is a solution of $\mathrm{Im}\,\tilde{\chi}(\omega) = 0$, since $f_M'(p)$ is an odd function of p, which makes the integrals vanish in equation (A.9). We now evaluate the integral of $\mathrm{Re}\,\tilde{\chi}(\omega)$. For $\omega_r > 0$, the first equation in (A.8) becomes [16]

$$\begin{aligned} \mathrm{Re}\,\tilde{\chi}(\omega) &= -\frac{1}{T}\left[1 - \sqrt{\pi}\, y\, \mathrm{e}^{y^2} \mathrm{erfc}(y)\right] \\ &\equiv -\frac{1}{T} f(y) \end{aligned} \tag{A.11}$$

with the scaled variable $y \equiv \omega_i \sqrt{M/2T}$, where

$$\mathrm{erfc}(y) = \frac{2}{\sqrt{\pi}} \int_y^{\infty} \mathrm{e}^{-t^2}\, \mathrm{d}t \tag{A.12}$$

is the complimentary error function. For $\omega_i = -|\omega_i| < 0$, it is straightforward to show that the last equation in (A.8) becomes

$$\begin{aligned} \mathrm{Re}\,\tilde{\chi}(\omega) &= -\frac{1}{T}\left[1 + \sqrt{\pi}|y|\mathrm{e}^{y^2}(2 - \mathrm{erfc}(|y|))\right] \\ &\equiv -\frac{1}{T} g(y). \end{aligned} \tag{A.13}$$

For $\omega_i = 0$, we have $\mathrm{Im}\, f_M'(\omega_r + \mathrm{i}\omega_i) = 0$ and the second equation in equation (A.8) simply reduces to $\mathrm{Re}\,\tilde{\chi}(\omega) = -1/T$. Note that $f(y)$ is a monotonically decreasing function of y, varying from unity to zero as y grows from zero to arbitrarily large values. On the other hand, $g(y)$ increases monotonically with y, from unity to arbitrarily large values.

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